



TITLE:

On Nunokawa's Lemma (Study on Non-Analytic and Univalent Functions and Applications)

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On Nunokawa's Lemma

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Abstract

I. S. Jack [1] proved the following lemma :
Let $w(z)$ be regular in the unit disc, $w(0) = 0$. Then if $w(z)$ attains its maximum value on the circle $|z| = r$ at a point z_1 , then we can write

$$z_1 w'(z_1) = k w(z_1)$$

where k is real and $1 \leq k$. Many and many mathematicians applied the above lemma and obtained numerous interesting results. In this paper, we will obtain a lemma which may be connected intimately to the Jack's lemma.

1 Basic geometrical property

Property 1 Let $\varphi(z)$ be analytic in $|z| < 1$, $\varphi(z) \neq 0$ in $|z| < 1$ and suppose that

$$\min_{|z| \leq r} |\varphi(z)| = |\varphi(z_0)|$$

and

$$\max_{|z| \leq r} |\varphi(z)| = |\varphi(z_1)|$$

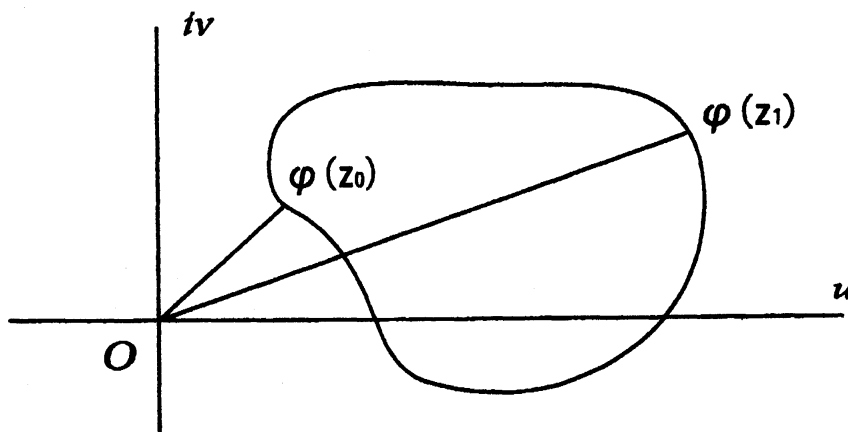
where $0 < r < 1$ and $|z_0| = |z_1| = r$. Then we have

$$\frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = \operatorname{Re} \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = \left(\frac{d \arg \varphi(z)}{d\theta} \right)_{\theta=\theta_0} < 0$$

and

$$\frac{z_1 \varphi'(z_1)}{\varphi(z_1)} = \operatorname{Re} \frac{z_1 \varphi'(z_1)}{\varphi(z_1)} = \left(\frac{d \arg \varphi(z)}{d\theta} \right)_{\theta=\theta_1} > 0$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $z_0 = re^{i\theta_0}$ and $z_1 = re^{i\theta_1}$.



A proof of Property 1 is trivial by considering geometrical property.

2 Nunokawa's lemma

Lemma 1 *Let $\varphi(z)$ be analytic in $|z| < 1$, $1 < \varphi(0)$ and suppose that there exists a point z_0 , $|z_0| < 1$ such that*

$$\begin{aligned} 1 < |\varphi(z)| & \quad \text{for } |z| < |z_0| \\ 1 = |\varphi(z_0)| & \quad \text{and } \varphi(z_0) \neq -1 \end{aligned}$$

or

$$\min_{|z| \leq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1 \neq -\varphi(z_0).$$

Then we have

$$\frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = \operatorname{Re} \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} \leq -\frac{\varphi(0) - 1}{\varphi(0) + 1}.$$

Proof. Let us put

$$(1) \quad \varphi(z) = \frac{1 + p(z)}{1 - p(z)} \quad \text{for } |z| < |z_0|.$$

Then it follows that

$$p(z) = \frac{\varphi(z) - 1}{\varphi(z) + 1}$$

and

$$0 < p(0) = \frac{\varphi(0) - 1}{\varphi(0) + 1} = \operatorname{Re} \frac{\varphi(0) - 1}{\varphi(0) + 1} < 1.$$

From the hypothesis of the Lemma 1 and (1), we have

$$0 < \operatorname{Re} p(z) \quad \text{for } |z| < |z_0|$$

and

$$0 = \operatorname{Re} p(z_0).$$

Putting

$$\Phi(z) = \frac{p(0) - p(z)}{p(0) + p(z)}, \quad \Phi(0) = 0$$

and applying the same method as the proof of [2], we have

$$|\Phi(z)| < 1 \quad \text{for } |z| < |z_0|$$

and

$$|\Phi(z_0)| = 1$$

and therefore

$$\frac{z_0 \Phi'(z_0)}{\Phi(z_0)} = \frac{-2p(0)z_0 p'(z_0)}{p(0)^2 - p(z_0)^2} = \frac{-2p(0)z_0 p'(z_0)}{p(0)^2 + |p(z_0)|^2} \geq 1.$$

It shows that

$$-z_0 p'(z_0) \geq \frac{1}{2} \left(\frac{p(0)^2 + |p(z_0)|^2}{p(0)} \right).$$

From (1) and Lemma 1, we have

$$\begin{aligned}
 \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} &= \operatorname{Re} \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} \\
 &= \frac{2z_0 p'(z_0)}{1 - p(z_0)^2} \\
 &= \frac{2z_0 p'(z_0)}{1 + |p(z_0)|^2} \\
 &\leq -\frac{1}{p(0)} \left(\frac{p(0)^2 + |p(z_0)|^2}{1 + |p(z_0)|^2} \right) \\
 &= -\frac{p(0)}{p(0)^2} \left(\frac{p(0)^2 + |p(z_0)|^2}{1 + |p(z_0)|^2} \right) \\
 &= -p(0) \left(\frac{p(0)^2 + |p(z_0)|^2}{p(0)^2 + p(0)^2 |p(z_0)|^2} \right) \\
 &< -p(0) \\
 &= -\left(\frac{\varphi(0) - 1}{\varphi(0) + 1} \right).
 \end{aligned}$$

It completes the proof. □

Applications of this lemma will be obtained by me and my friends.

References

- [1] I. S. Jack, *Functions starlike and convex of order α* , J. London Math. Soc. (2), **3** (1971), 469-474.
- [2] M. Nunokawa, *On properties of Non-Carathéodory functions*, Proc. Japan Acad. Vol. 68, Ser. A, No.6 (1992), 152-153.

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